

# Dimension Reduction Near Periodic Orbits of Hybrid Systems: Appendix

*Sam Burden  
Shai Revzen  
S. Shankar Sastry*



Electrical Engineering and Computer Sciences  
University of California at Berkeley

Technical Report No. UCB/EECS-2011-100

<http://www.eecs.berkeley.edu/Pubs/TechRpts/2011/EECS-2011-100.html>

September 7, 2011

Report Documentation Page				Form Approved OMB No. 0704-0188	
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.					
1. REPORT DATE <b>07 SEP 2011</b>		2. REPORT TYPE		3. DATES COVERED <b>00-00-2011 to 00-00-2011</b>	
4. TITLE AND SUBTITLE <b>Dimension Reduction Near Periodic Orbits of Hybrid Systems: Appendix</b>				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S)				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) <b>University of California, Berkely,Electrical Engineering and Computer Sciences,Berkeley,CA,94720</b>				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S)	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION/AVAILABILITY STATEMENT <b>Approved for public release; distribution unlimited</b>					
13. SUPPLEMENTARY NOTES					
14. ABSTRACT <b>When the Poincaré map associated with a periodic orbit of a hybrid dynamical system has constant-rank iterates we demonstrate the existence of a constant-dimensional invariant subsystem near the orbit which attracts all nearby trajectories in finite time. This result shows that the longterm behavior of a hybrid model with a large number of degrees-of-freedom may be governed by a low-dimensional smooth dynamical system. The appearance of such simplified models enables the translation of analytical tools from smooth systems?such as Floquet theory?to the hybrid setting and provides a bridge between the efforts of biologists and engineers studying legged locomotion.</b>					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT <b>Same as Report (SAR)</b>	18. NUMBER OF PAGES <b>9</b>	19a. NAME OF RESPONSIBLE PERSON
a. REPORT <b>unclassified</b>	b. ABSTRACT <b>unclassified</b>	c. THIS PAGE <b>unclassified</b>			

Copyright © 2011, by the author(s).  
All rights reserved.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission.

#### Acknowledgement

We thank Saurabh Amin, Jonathan Glidden, Humberto Gonzalez, John Guckenheimer, and Ramanarayan Vasudevan for helpful conversations and careful readings of this paper.

S. Burden was supported in part by an NSF Graduate Research Fellowship. S. Revzen was supported in part by NSF Frontiers for Integrative Biology Research (FIBR), Grant No. 0425878-Neuromechanical Systems Biology. Part of this research was sponsored by the Army Research Laboratory under Cooperative Agreements W911NF-08-2-0004 and W911NF-10-2-0016.

# Dimension Reduction Near Periodic Orbits of Hybrid Systems: Appendix

Samuel Burden, Shai Revzen and S. Shankar Sastry

**Abstract**—When the Poincaré map associated with a periodic orbit of a hybrid dynamical system has constant-rank iterates, we demonstrate the existence of a constant-dimensional invariant subsystem near the orbit which attracts all nearby trajectories in finite time. This result shows that the long-term behavior of a hybrid model with a large number of degrees-of-freedom may be governed by a low-dimensional smooth dynamical system. The appearance of such simplified models enables the translation of analytical tools from smooth systems—such as Floquet theory—to the hybrid setting and provides a bridge between the efforts of biologists and engineers studying legged locomotion.

## I. INTRODUCTION

Dynamic multi-legged locomotion presents a daunting control task. A large number of degrees-of-freedom (DOF) must be rapidly and precisely coordinated in the face of state and environmental uncertainty. The ability of individual limbs to exert forces on the body varies intermittently with ground contact, body posture, and the efforts of other appendages. Finally, the motion itself affects sensor measurements, complicating pose estimation. In spite of these difficulties, animals at all levels of complexity have mastered the art of rapid legged locomotion over complex terrain at speeds far exceeding those of comparable robotic platforms [1], [2], [3], [4].

Numerous architectures have been proposed to explain how animals control their limbs. For steady-state locomotion, most posit a principle of coordination, synergy, symmetry or synchronization, and there is a surfeit of neurophysiological data to support these hypotheses [5], [6], [7], [8]. In effect, the large number of DOF available to an animal are collapsed during regular motion to a low-dimensional dynamical attractor that may be captured by a *template* model embedded within a higher-dimensional model *anchored* to the animal’s morphology [9], [10]. In this view, only a few parameters like frequency and coupling strength are required to describe the dynamics of any particular periodic gait over a broad range of animal morphologies, offering a tantalizing target for experimental biologists. Were the dynamics of legged animals smooth as a function of position and momentum, Floquet theory [11] provides a canonical form for the structure of the stability basin of a limit cycle [12], [13]. In such a canonical form, the template may appear as an invariant attractor of the linearized dynamics and be amenable to quantitative

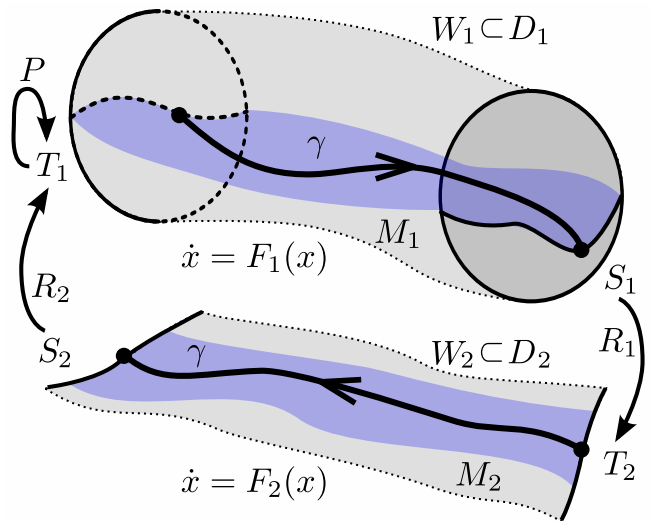


Fig. 1. Illustration of the main result in a two-domain system: Whenever iterates of the Poincaré map  $P$  associated with a periodic orbit  $\gamma$  of a hybrid dynamical system are constant-rank, there is a constant-dimensional smooth submanifold  $M_j$  in each domain  $D_j$  which is invariant under the flow of the vector field  $F_j$  and which attracts all trajectories starting in an open set  $W_j \subset D_j$  containing  $\gamma$  in finite time.

measurement [14], [15]. A substantial motivation for the present work has been to provide a theoretical framework for applying this empirical approach to study legged locomotion. The dynamics of legged locomotion are rarely smooth due to intermittent contact of limbs with the substrate, so we have generalized this approach to be applicable to a class of non-smooth systems called *hybrid dynamical systems*.

We relegate a formal definition of the class of hybrid systems under consideration to Section III. Informally, hybrid dynamical systems are comprised of differential equations written over disparate domains together with rules for switching between the domains. Of particular interest to us are periodic orbits of such systems. From a modeling viewpoint, a stable hybrid periodic orbit provides a natural abstraction for the dynamics of steady-state legged locomotion. This approach has been widely adopted, generating a variety of models of bipedal [16], [17], [18] and multi-legged [19], [20] locomotion as well as some general control-theoretic techniques for composition [21], coordination [22], and stabilization [23], [24], [25] of such models. In certain cases, it has been possible to formally embed a low-dimensional abstraction in a higher-dimensional physically-realistic model [26], [27].

This paper provides a conceptual link between formal analysis of hybrid periodic orbits and the dramatic dimension

S. Burden and S. Sastry are with the Department of Electrical Engineering and Computer Sciences, University of California at Berkeley, CA, USA [sburden](mailto:sburden@eecs.berkeley.edu), [sastry@eecs.berkeley.edu](mailto:sastry@eecs.berkeley.edu)

S. Revzen is with the Department of Electrical and Systems Engineering, University of Pennsylvania, Philadelphia, PA, USA [hrevzen@seas.upenn.edu](mailto:hrevzen@seas.upenn.edu)

reduction observed empirically in successful legged locomotors. Under the condition that iterates of the Poincaré map associated with a periodic orbit are constant rank, we demonstrate the existence of a constant-dimensional invariant subsystem which attracts all nearby trajectories in finite time. Analogous results for smooth dynamical systems typically impose stringent assumptions on the dynamics such as exact symmetries (cf. §8.9 in [28]) or timescale separation (cf. Chapter 4 in [13]). In contrast, the results of this paper imply that hybrid dynamical systems may exhibit dimension reduction near periodic orbits solely due to the interaction of the switching dynamics with the smooth flow.

## Organization

The hybrid systems we consider are constructed using switching maps defined between boundaries of smooth dynamical systems. The behavior of such systems can be studied by alternately applying flows and maps. Thus, we begin in Section II by developing several results which provide canonical forms for the behavior of flows and maps near periodic orbits and fixed points, respectively. Then, we define hybrid systems in Section III and use these results to characterize the dynamics near their periodic orbits. Examples are presented in Section IV and implications of the results for the design and analysis of legged locomotors are explored in Section V.

## II. SMOOTH DYNAMICAL SYSTEMS

This section contains three technical results used in the proof of the Theorem of Section III. The first two results concern smooth dynamical systems<sup>1</sup> and may be found in textbooks, hence we state them without proof. The third establishes, under a non-degeneracy condition, a canonical form for the invariant set of a smooth map near a fixed point. A reader interested in the main result of this paper may proceed to Section III and refer to this section as needed.

### A. Differential Geometry

We assume familiarity with the tools and terminology of differential geometry. If any of the concepts we discuss are unfamiliar, we refer the reader to [28], [29] for more details.

**Definition 1.** A smooth dynamical system is a pair  $(M, G)$ :  
 $M$  is a smooth manifold with boundary  $\partial M$ ;  
 $G$  is a smooth vector field on  $M$ , i.e.  $G \in \mathcal{T}(M)$ .

### B. Flows Between Surfaces

We review the fact that the flow near a trajectory passing transversally between two surfaces has a simple form (cf. Chapter 11.2 in [30]). In particular, nearby trajectories can be obtained from an embedding of a product manifold. This will be the prototype for the dynamics near a periodic orbit in one domain of a hybrid system.

<sup>1</sup>For notational convenience, we work with objects which possess continuous derivatives of all orders. However, the results in this paper are valid if we only assume continuous differentiability.

**Lemma 1.** Let  $(M, G)$  be a smooth dynamical system and  $\phi : \mathcal{F} \rightarrow M$  its maximal flow. Suppose  $T, S \subset M$  are smooth embedded submanifolds,  $S$  has codimension 1,  $\phi(\alpha, \xi) \in S$  for some  $\alpha > 0$  and  $\xi \in T$  where  $G$  is transverse to  $T$  at  $\xi$  and  $S$  at  $\phi(\alpha, \xi)$ . Then we have the following consequences:

- (i) there is a neighborhood  $U \subset M$  containing  $\xi$  and a smooth map  $\eta : U \rightarrow \mathbb{R}$  so that  $\eta(\xi) = \alpha$  and for all  $x \in U$ ,  $\eta(x) > 0$  and  $\phi(\eta(x), x) \in S$ ;  $\eta$  is called the time-to-impact map;
- (ii) with  $V := U \cap T$ , the map  $\psi : [0, 1] \times V \rightarrow M$  with

$$\psi(\sigma, v) := \phi(\eta(v)\sigma, v)$$

is a smooth embedding into  $M$  whose image contains the trajectory  $\gamma = \{\phi(t, \xi) : 0 \leq t \leq \alpha\}$ .

**Remark 1.** This lemma is applicable when  $T, S \subset \partial M$ , which will be relevant in the study of hybrid systems.

### C. Gluing Flows

In this section, we provide a method for gluing two smooth dynamical systems together along their boundaries to obtain a new smooth system; this construction uses basic results from differential topology (cf. Theorem 8.2.1 in [31]). We will use this construction in Section III to attach distinct hybrid domains to one another.

**Lemma 2.** Suppose  $(M_1, G_1), (M_2, G_2)$  are smooth  $n$ -dimensional dynamical systems,  $\varphi : \partial M_1 \rightarrow \partial M_2$  is a diffeomorphism,  $G_1$  is outward-pointing along  $\partial M_1$  and  $G_2$  is inward-pointing along  $\partial M_2$ . Then the topological quotient  $M := \frac{M_1 \amalg M_2}{\partial M_1 \simeq \partial M_2}$  can be made into a smooth manifold for which (i) the inclusions  $M_j \hookrightarrow M$  are smooth embeddings and (ii) there is a smooth vector field  $G \in \mathcal{T}(M)$  that restricts to  $G_j$  on  $M_j$ ,  $j = 1, 2$ .

**Remark 2.** The smooth structure described in Lemma 2 is unique. Further, if  $\tilde{G}_1$  and  $\tilde{G}_2$  are any other smooth vector fields on  $M_1$  and  $M_2$  which satisfy the hypotheses of the Lemma, the corresponding quotient  $\tilde{M}$  is diffeomorphic to  $M$  (cf. Chapter 8 in [31]).

### D. Invariant Set of a Smooth Map Near a Fixed Point

In studying hybrid dynamical systems, we encounter smooth maps  $f : M \rightarrow M$  which are not diffeomorphisms. Viewing iteration of  $f$  as a discrete dynamical system, we wish to study the behavior of these iterates near a fixed point  $f(\xi) = \xi$ . Note that if  $f$  has constant rank equal to  $k \in \mathbb{N}$ , then its image  $f(M) \subset M$  is an embedded  $k$ -dimensional submanifold near  $\xi$  by the Rank Theorem (cf. Theorem 7.13 in [29]). With an eye toward dimension reduction, one might hope that the composition  $(f \circ f) : M \rightarrow f(M)$  is also constant-rank, but this is not generally true<sup>2</sup>. If it is true that iterates of  $f$  are eventually constant-rank near the fixed point  $\xi$ , then one can study the behavior of these iterates by restricting the domain to a lower-dimensional submanifold.

<sup>2</sup>Consider the map  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $f(x, y) := (x^2, x)$ .

**Lemma 3.** *Let  $M$  be a smooth manifold,  $f : M \rightarrow M$  a smooth map with  $f(\xi) = \xi$  for some  $\xi \in M$ , suppose the rank of  $f$  is bounded above by  $n \in \mathbb{N}$ , and suppose the composition of  $f$  with itself  $n$  times,  $f^n$ , has constant rank equal to  $r \in \mathbb{N}$  on a neighborhood of  $\xi$ . Then  $f^n(M)$  is an  $r$ -dimensional embedded submanifold near  $\xi$  and there are neighborhoods  $U, V \subset f^n(M)$  containing  $\xi$  for which  $f$  maps  $U$  diffeomorphically onto  $V$ .*

In the proof of Lemma 3, we make use of an elementary fact from linear algebra. The result is easily obtained by passing to the Jordan form.

**Proposition 1.** *If  $A \in \mathbb{R}^{m \times m}$  and  $\text{rank } A \leq n$ , then  $\text{rank}(A^{2n}) = \text{rank}(A^n)$ .*

*Proof.* (of Lemma 3) By the Rank Theorem (cf. Theorem 7.13 in [29]), there is a neighborhood  $N \subset M$  of  $\xi$  for which  $\Sigma := f^n(N)$  is an  $r$ -dimensional embedded submanifold and by Proposition 1 we have

$$\begin{aligned} \text{rank}(f^n|_{\Sigma})_*(\xi) &= \text{rank}(f^n \circ f^n)_*(\xi) \\ &= \text{rank } f^n_*(\xi). \end{aligned}$$

Therefore  $(f^n|_{\Sigma})_* : T_{\xi}\Sigma \rightarrow T_{\xi}\Sigma$  is a bijection, so by the Inverse Function Theorem (cf. Theorem 7.10 in [29]), there is a neighborhood  $W \subset \Sigma$  containing  $\xi$  so that  $f^n(W) \subset \Sigma$  and  $f^n|_W : W \rightarrow f^n(W)$  is a diffeomorphism.

By continuity of  $f$ , there is a neighborhood  $L \subset N$  containing  $\xi$  for which  $f(L) \subset N$  and  $f^n(L) \subset W$ . The set  $U := f^n(L)$  is a neighborhood of  $\xi$  in  $\Sigma$ . Further, we have

$$f(U) = f \circ f^n(L) = f^n \circ f(L) \subset \Sigma.$$

The restriction  $f^n|_U : U \rightarrow f^n(U)$  is a diffeomorphism since  $U \subset W$ , whence  $f|_U$  is a diffeomorphism onto its image,  $V := f(U) \subset \Sigma$ .  $\square$

### III. HYBRID DYNAMICAL SYSTEMS

We describe a class of hybrid systems useful for modeling legged locomotion, then restrict our attention to the behavior of such systems near periodic orbits. It was shown in [32] that the Poincaré map of a hybrid system is generally not full rank. We explore the geometric consequences of this rank loss and demonstrate, under a non-degeneracy condition, the existence of a smooth invariant subsystem which attracts all nearby trajectories in finite time.

#### A. Hybrid Differential Geometry

For our purposes, it is expedient to define hybrid dynamical systems over disjoint unions of smooth manifolds.

**Definition 2.** *A smooth hybrid manifold is a finite disjoint union of connected smooth manifolds  $M = \coprod_{j \in J} M_j$ .*

**Remark 3.** *The dimensions of the constituent manifolds are not required to be equal.*

Differential geometric constructions which are confined to a single manifold have natural generalizations to such spaces, and we will prepend the modifier “hybrid” to make it

clear when this generalization is being invoked. For instance, the *hybrid tangent bundle*  $TM$  is the disjoint union of the tangent bundles  $TM_j$ , the *hybrid boundary*  $\partial M$  is the disjoint union of the boundaries  $\partial M_j$ , and a *hybrid open set*  $U \subset M$  is obtained from a disjoint union of open sets  $U_j \subset M_j$ . Generalizing maps between manifolds requires more care, hence we provide explicit definitions.

**Assumption 1.** *To simplify the exposition, we henceforth assume all manifolds and maps between manifolds are smooth.*

**Definition 3.** A hybrid map

$$f : \coprod_{j \in J} M_j \rightarrow \coprod_{\ell \in L} N_{\ell}$$

between hybrid manifolds restricts to a map  $f|_{M_j} : M_j \rightarrow N_{\ell}$ , some  $\ell \in L$ , for each  $j \in J$ . The hybrid map is called constant-rank, injective, or surjective if each  $f|_{M_j}$  is as well. It is called an embedding if each  $f|_{M_j}$  is an embedding and  $f$  is a homeomorphism onto its image.

**Definition 4.** The hybrid pushforward  $f_* : TM \rightarrow TN$  is the hybrid map defined piecewise as  $f_*|_{TM_j} := (f|_{M_j})_*$ .

**Definition 5.** A hybrid vector field on a hybrid manifold  $M := \coprod_{j \in J} M_j$  is a hybrid map  $G : M \rightarrow TM$  for which  $G|_{M_j}$  is a vector field on  $M_j$ , i.e.  $G|_{M_j} \in \mathcal{T}(M_j)$ . We let  $\mathcal{T}(M)$  denote the space of hybrid vector fields on  $M$ .

To state the main result of this paper, we need to embed manifolds into hybrid manifolds. This can be achieved by first partitioning the smooth manifold to obtain a hybrid manifold, then embedding this hybrid manifold via the previous definitions.

**Definition 6.** A partition of an  $n$ -dimensional manifold  $M$  is a finite set  $\{M_j\}_{j \in J}$  of embedded  $n$ -dimensional submanifolds  $M_j \subset M$  for which  $\bigcup_{j \in J} M_j = M$  and if  $i \neq j$  we have  $\text{Int}(M_j) \cap \text{Int}(M_i) = \emptyset$ .

**Definition 7.** A hybrid embedding of a manifold  $M$  into a hybrid manifold  $N := \coprod_{j \in J} N_j$  is determined by a partition  $\{M_j\}_{j \in J}$  of  $M$  and a hybrid embedding

$$f : \coprod_{j \in J} M_j \rightarrow \coprod_{j \in J} N_j$$

for which  $f_j : M_j \rightarrow N_j$  for each  $j \in J$ . Any  $G \in \mathcal{T}(M)$  may be pushed forward to a unique  $f_*G \in \mathcal{T}(f(M))$ . The image of  $f$  is a hybrid embedded submanifold.

With these preliminaries established, we can define the class of hybrid systems considered in this paper.

**Definition 8.** A hybrid dynamical system is specified by a triple  $H := (D, F, R)$  where:

$D = \coprod_{j \in J} D_j$  is a hybrid manifold;

$F \in \mathcal{T}(D)$  is a hybrid vector field on  $D$ ;

$R : S \rightarrow T$  is a hybrid map,  $S, T \subset \partial D$  are hybrid embedded submanifolds, and  $S$  has codimension 1.

As in [33], we call  $R$  the reset map and  $S$  the guard.

Note that if  $F$  is tangent to  $S$  at  $x \in D$ , there is a possible ambiguity in determining a trajectory from  $x$ —one may either follow the flow of  $F$  on  $D$  or apply the reset map to obtain a new initial condition  $y = R(x)$ .

**Assumption 2.** *To ensure that trajectories are uniquely defined, we assume that  $F$  is outward-pointing on  $S$ .*

To the best of our knowledge, this definition of a hybrid dynamical system has not appeared before. However, in light of the constructions contained in this section, it may be seen as a mild generalization of a *simple hybrid system* (cf. §3.2 in [33]). Further, in Section III-C we will see that this definition supports powerful geometric analysis of the dynamics near a hybrid periodic orbit. Finally, this class of hybrid systems encompasses many closed-loop models of legged locomotion [16], [17], [18], [19], [20], [23], [24], [26], [27]. We contend that these facts justify the introduction of the novel definition.

**Remark 4.** *As defined above, hybrid dynamical systems possess unique executions or trajectories from every initial condition. This fact can be demonstrated algorithmically. For any  $x \in D_j$ , obtain the maximal integral curve of  $F|_{D_j}$ . This integral curve must either: a) continue for all time; b) exit  $D_j$  without intersecting  $\partial D_j$  (in which case execution terminates); or c) intersect the boundary at  $y \in \partial D_j$ . If  $y \in S$ , the map  $R$  is applied to obtain a new initial condition  $R(y) \in T$ , and otherwise execution terminates.*

The following definition enables us to embed smooth dynamical systems into hybrid dynamical systems in such a way that trajectories of the smooth system are preserved in the hybrid system. We illustrate the use of this construction by giving a terse description of trajectories for this class of hybrid systems. In the subsequent sections, we use this construction to state the main results of this paper.

**Definition 9.** *A hybrid dynamical embedding of a dynamical system  $(M, G)$  into a hybrid dynamical system  $(D, F, R)$  is a hybrid embedding  $f : M \rightarrow D$  for which  $f_*G = F|_{f(M)}$  and  $R|_{f(M) \cap S}$  is a hybrid diffeomorphism from  $f(M) \cap S$  onto  $f(M) \cap T$ .*

**Remark 5.** *A trajectory of a hybrid dynamical system  $H$  may be obtained from a hybrid dynamical embedding of the system  $(J, \frac{\partial}{\partial t})$ , where  $J \subset \mathbb{R}$  is a connected interval.*

**Definition 10.** *A  $\tau$ -periodic orbit of a hybrid dynamical system is a hybrid dynamical embedding  $\gamma$  of the dynamical system  $(S^1, \frac{2\pi}{\tau} \frac{\partial}{\partial \varphi})$ , where  $S^1$  is the unit circle.*

**Remark 6.** *We alternately refer to  $\gamma$  as a periodic trajectory and often write  $\gamma$  in place of the image  $\gamma(S^1)$ .*

## B. Hybrid Poincaré map

To state the main result of this paper, we must construct the Poincaré map associated with a periodic orbit of a hybrid system. This has been developed before [24], [32]; the construction is more delicate than for smooth systems since trajectories of hybrid systems do not necessarily vary

continuously with initial conditions. We directly demonstrate this continuous dependence in the construction of the map.

Let  $H = (D, F, R)$  be a hybrid dynamical system and  $\gamma$  a periodic orbit of  $H$  with period  $\tau$ . Then  $\gamma$  undergoes a finite number of transitions  $k \in \mathbb{N}$ , so we may index the corresponding sequence of domains as<sup>3</sup>  $D_1, \dots, D_k$ . Without loss of generality, assume the  $D_j$ 's are distinct<sup>4</sup>; let  $\gamma_j := \gamma \cap T_j$  be the entry point of  $\gamma$  in  $D_j$  and let  $\tau_j$  be the time spent by  $\gamma$  in  $D_j$ . We wish to construct the Poincaré map  $P$  associated with  $\gamma$  over a neighborhood of  $\gamma_j$  in  $T$ . To do this, we must ensure that each initial condition in that neighborhood has a well-defined non-zero first-return time to  $T$ ; the following assumption guarantees this.

**Assumption 3.** *To ensure the Poincaré map is well-defined, we assume  $F$  is transverse to  $T$  and not outward-pointing.*

Now for  $j = 1, \dots, k$  and referring to Fig. 2 for an illustration of these objects, let:

- $\phi_j : \mathcal{F}_j \rightarrow D_j$  be the maximal flow of  $F$  on  $D_j$ ;
- $T_j \subset T \cap D_j$  be a neighborhood of  $\gamma_j$  over which Lemma 1 may be applied between  $T$  and  $S$  on  $D_j$ ;
- $\psi_j : [0, 1] \times T_j \rightarrow D_j$  be the embedding from Lemma 1;
- $S_j := \psi_j(1, T_j) \subset S \cap D_j$  be the image of  $T_j$  in  $S$  under the flow on  $D_j$ ;
- $R_j : S_j \rightarrow T$  denote the restriction  $R_j := R|_{S_j}$ ;
- $p_j : T_j \rightarrow T$  be defined by  $p_j(u) := R_j(\psi_j(1, u))$ .

The Poincaré map over the section  $T_j$  is obtained formally by iterating the  $p$ 's around the cycle:

$$P_j := p_{j-1} \circ \dots \circ p_1 \circ p_k \circ \dots \circ p_j. \quad (1)$$

The neighborhood  $\Sigma_j \subset T_j$  of  $\gamma_j$  over which this map is well-defined is determined by pulling  $T_j$  backward around the cycle,

$$\Sigma_j = (p_j^{-1} \circ \dots \circ p_k^{-1} \circ p_1^{-1} \circ \dots \circ p_{j-1}^{-1})(T_j),$$

and similarly for any iterate of  $P_j$ .

It is a standard result for smooth dynamical systems that *Floquet multipliers* (the eigenvalues of the linearized Poincaré map) do not depend on the choice of Poincaré section (cf. Section 1.5 in [13]). The following lemma generalizes this result to the hybrid setting by demonstrating that if a Poincaré map obtained from one domain has an attracting invariant submanifold via Lemma 3, then the map obtained in any other starting domain has a diffeomorphic attracting submanifold. As a consequence, non-zero Floquet multipliers are shared between the  $P_j$ 's after a sufficient number of iterations.

**Lemma 4.** *Let  $j \in \{1, \dots, k\}$  and  $n \geq \min_\ell \dim D_\ell$ . If  $P_j^n$  has constant rank equal to  $r$  near  $\gamma_j$ , then  $P_\ell^{n+1}$  has constant rank equal to  $r$  near  $\gamma_\ell$  for all  $\ell \in \{1, \dots, k\}$ .*

In the proof of Lemma 4, we make use of an elementary fact from linear algebra. The result is easily obtained from Sylvester's inequality (cf. Appendix A.5.4 in [34]).

<sup>3</sup>We regard subscripts modulo  $k$  so that  $D_k \equiv D_0$ .

<sup>4</sup>Otherwise we can find  $\{B_j\}_{j=1}^k$  be such that  $B_j \subset D_j$  is open,  $\gamma \subset \bigcup_{j=1}^k B_j$ , and  $B_i \cap B_j = \emptyset$  if  $i \neq j$ , then proceed on  $D := \bigcup_{j=1}^k B_j$ .



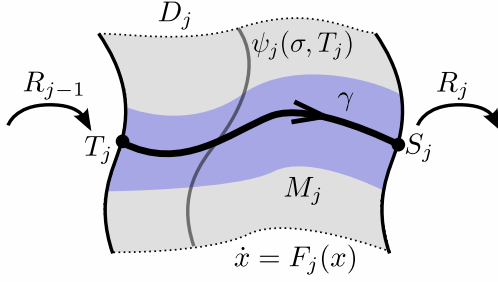


Fig. 2. Illustration of a hybrid domain containing part of a periodic orbit. The periodic orbit  $\gamma$  enters the domain  $D_j$  inside the submanifold  $T_j \subset \partial D_j$  via the reset map  $R_{j-1}$ . Initial conditions in  $T_j$  flow to the submanifold  $S_j \subset \partial D_j$  via the vector field  $F_j$ . The map  $\psi_j$  smoothly reparameterizes these trajectories by the time required to pass from  $T_j$  to  $S_j$ , thus  $\psi_j(\sigma, T_j)$  is an embedded submanifold for all  $\sigma \in [0, 1]$  and  $S_j = \psi_j(1, T_j)$ . While in domain  $D_j$ ,  $\gamma$  lies in the invariant submanifold  $M_j$  constructed in Theorem 1. By construction,  $M_j$  is an integral submanifold of  $F_j$  and  $\dim M_j \leq \dim D_j$ ; see Fig. 1 for an illustration when  $\dim M_j < \dim D_j$ .

**Proposition 2.** For  $j \in \{1, \dots, k\}$ , suppose  $a_j \in \mathbb{R}^{n_{j+1} \times n_j}$  where  $n_k = n_1$ , define  $A_j := a_{j-1} \cdots a_1 a_k \cdots a_j$ , and let  $n \geq \min_j n_j$ . Then for all  $\ell \in \{1, \dots, k\}$ , we have  $\text{rank } A_\ell^{n+1} = \text{rank } A_j^{n+1} = \text{rank } A_j^n$ .

*Proof.* (of Lemma 4) By Lemma 3, there is a neighborhood  $N_j \subset T_j$  of  $\gamma_j$  on which  $P_j^n$  has constant rank equal to  $r$ . Fix  $\ell \in \{1, \dots, k\}$ , let  $p_\ell^j := p_{j-1} \circ \cdots \circ p_\ell$ , and define  $N_\ell := (p_\ell^j)^{-1}(N_j)$ . Then  $N_\ell \subset T_\ell$  is a neighborhood of  $\gamma_\ell$  and, for all  $x \in N_\ell$ , by Sylvester's inequality

$$\text{rank}(P_\ell^{n+1})_*(x) \leq \text{rank}(P_j^n)_*(p_\ell^j(x)) = r.$$

Furthermore by Proposition 2,

$$\begin{aligned} \text{rank}(P_\ell^{n+1})_*(\gamma_\ell) &= \text{rank}(P_j^{n+1})_*(\gamma_j) \\ &= \text{rank}(P_j^n)_*(\gamma_j) = r. \end{aligned}$$

We conclude the rank of  $P_\ell^{n+1}$  is at least  $r$  on a neighborhood  $L_\ell \subset T_\ell$  of  $\gamma_\ell$ , whence  $\text{rank } P_\ell^{n+1} = r$  on  $L_\ell \cap N_\ell$ .  $\square$

As a consequence, if the Poincaré map associated with any section for the periodic orbit  $\gamma$  satisfies the hypotheses of Lemma 3, then the Poincaré map associated with any other section also satisfies the hypotheses.

**Remark 7.** It may be easier to evaluate the rank of the Poincaré map in some domains than others. In particular, if  $P_j$  is a diffeomorphism for some  $j \in \{1, \dots, k\}$ , then all iterates are constant rank.

### C. Hybrid Invariant Subsystem

This section contains the main result of this paper: when iterates of the Poincaré map associated with a periodic orbit of a hybrid dynamical system have constant rank, trajectories starting near the orbit converge in finite time to an embedded smooth dynamical system.

**Theorem 1.** Let  $H = (D, F, R)$  be a hybrid dynamical system,  $\gamma$  a periodic orbit of  $H$ , and suppose the composition of any Poincaré map for  $\gamma$  with itself at least  $\min_j \dim D_j$

times has constant rank equal to  $r$  on a neighborhood of its fixed point. Then there is an  $(r+1)$ -dimensional dynamical system  $(M, G)$ , a hybrid dynamical embedding  $f : M \rightarrow D$ , and an open hybrid set  $W \subset D$  so that  $\gamma \subset f(M) \cap W$  and trajectories starting in  $W$  flow into  $M$  in finite time.

*Proof.* By assumption, we may apply Lemma 3 to  $P$  to obtain a neighborhood  $N \subset T_1$  of  $\gamma \cap T_1$ , an embedded submanifold  $\Sigma \subset T_1$  containing  $\gamma \cap T_1$ , and a pair of neighborhoods  $U, V \subset \Sigma$  of  $\gamma \cap T_1$  so that  $P|_U : U \rightarrow V$  is a diffeomorphism and  $\dim U = r$ . Now we consider the subset of  $D$  obtained by propagating each  $x \in U$  around one cycle. Let  $U_1 = U$  and  $U_j = P_{j-1}(U_{j-1})$  for  $j = 2, \dots, k$ . Away from the boundaries, we can obtain the desired set directly from  $\psi_j$  as  $\text{Int}(M_j) := \psi_j((0, 1), U_j)$ . For  $j = 2, \dots, k-1$  we can simply attach the corresponding boundaries to obtain  $M_j := \psi_j([0, 1], U_j)$ . However, since we may not assume  $U \subset V$  or  $V \subset U$  (only that  $U \cap V$  is a neighborhood of  $\gamma \cap T_1$ ), we must be careful in attaching the boundary between  $M_k$  and  $M_1$ . Thus, we let  $M_1 = \psi_1((0, 1), U_1) \cup (U_1 \cap P_k(U_k))$  and  $M_k = \psi_k([0, 1], U_k) \cup (\psi_k(1, U_k) \cap R_k^{-1}(U_1))$ . With this construction, for each  $j = 1, \dots, k$  we have that  $M_j$  is a smooth submanifold with boundary  $\partial M_j \subset T_j \cup S_j$  and  $\partial M_j$  contains both points in  $\gamma_j \cap \partial D_j$ ; see Fig. 2 for an illustration of  $M_j$ .

Since  $M_j$  is an integral submanifold of  $F$  on  $D_j$ , the vector field  $F$  restricts to  $M_j$ . Letting  $G_j$  denote this restriction, each  $(M_j, G_j)$  is a smooth dynamical system and  $G_j$  points inward on  $\partial M_j \cap T_j$  and outward on  $\partial M_j \cap S_j$ . Since  $P|_U$  is a diffeomorphism, each  $R_j|_{\partial M_j \cap S_j} : \partial M_j \cap S_j \rightarrow \partial M_{j+1} \cap T_{j+1}$  is a diffeomorphism as well. Therefore we may glue these systems together one-by-one via Lemma 2 to obtain a smooth dynamical system without boundary  $(M, G)$  which embeds into  $H$  and contains  $\gamma$ .

Finally, let  $N, U$ , and  $V$  be as above and let  $\delta > 0$  be an arbitrary positive number. Note that by continuity of  $P$  and the time-to-impact maps of Lemma 1, there is a neighborhood  $W_1 \subset N \subset T_1$  of  $\gamma \cap T_1$  so that  $P^n(W_1) \subset U \cap V$  and each  $w \in W_1$  flows into  $U \cap V$  before time  $n\tau + \delta$ . Since the  $P_j$ 's are continuous, for  $j = 2, \dots, k$  there are neighborhoods  $W_j \subset T_j$  of  $\gamma \cap T_j$  so that every  $w \in W_j$  flows into  $W_{j+1}$  before time  $\tau_j + \delta/k$ . Taking the union of these neighborhoods as  $W = \bigcup_{j=1}^k \psi_j([0, 1], W_j)$  yields an open hybrid submanifold  $W \subset D$  so that  $\gamma \subset M \cap W$  and every point in  $W$  flows into  $W_1$  before time  $\tau + \delta$ , and hence into  $M$  before time  $(n+1)\tau + 2\delta$ ; see Fig. 1 for an illustration of these neighborhoods in a particular two-domain hybrid dynamical system.  $\square$

**Corollary 1.**  $\gamma$  is asymptotically stable for  $H = (D, F, R)$  if and only if  $\gamma$  is asymptotically stable for  $(M, G)$ .

*Proof.* Since all trajectories in a neighborhood  $W$  of  $\gamma$  reach  $M$  in finite time and the hybrid flow is continuous near  $\gamma$ , trajectories in  $W$  will converge to  $\gamma$  asymptotically if and only if trajectories in  $W \cap M$  converge to  $\gamma$  asymptotically. This occurs precisely when  $\gamma$  is asymptotically stable for  $(M, G)$  since by construction  $M$  is an integral submanifold



of  $F$  and  $R|_{M \cap S} : M \cap S \rightarrow M \cap T$  is a diffeomorphism.  $\square$

If each of the  $D_j$ 's have the same dimension and  $R : S \rightarrow T$  is a diffeomorphism, the rank condition of Theorem 1 is trivially satisfied, and we can globalize the construction using Lemma 2. This provides a smooth  $n$ -dimensional generalization of the construction in [35].

**Corollary 2.** *Let  $H = (D, F, R)$  be a hybrid dynamical system with  $D = \coprod_{j \in J} D_j$ ,  $R : S \rightarrow T$ , and  $\partial D = S \cup T$ . If  $\dim D_j = n$  for all  $j \in J$  and  $R$  is a diffeomorphism, then there is a surjective hybrid dynamical embedding from an  $n$ -dimensional dynamical system  $(M, G)$  onto  $H$ .*

#### IV. EXAMPLE

##### A. Hybrid Floquet Coordinates

The following single-domain system clearly satisfies the hypotheses of Theorem 1, and demonstrates the canonical form for hybrid Floquet coordinates.

**Example 1.** *Let  $H = (D, F, R)$  be a hybrid system over the single domain  $D = [0, 1] \times \mathbb{R}^k \times \mathbb{R}^\ell$  with vector field  $F(t, x, z) = \frac{\partial}{\partial t} + \sum_{j=1}^k f^j(t, x) \frac{\partial}{\partial x^j} + \sum_{i=1}^\ell g^i(t, z^i) \frac{\partial}{\partial z^i}$ , reset map  $R : \{1\} \times \mathbb{R}^k \times \mathbb{R}^\ell \rightarrow \{0\} \times \mathbb{R}^k \times \mathbb{R}^\ell$  defined by  $R(1, x, z) = (0, x, Az)$  where  $A \in \mathbb{R}^{\ell \times \ell}$  is nilpotent,  $f(t, \xi) = 0$  for all  $t \in [0, 1]$  and some  $\xi \in \mathbb{R}^k$ , and  $g^i(t, 0) = 0$  for all  $i$ . Consider the Poincaré map*

$$P : \{0\} \times \mathbb{R}^k \times \mathbb{R}^\ell \rightarrow \{0\} \times \mathbb{R}^k \times \mathbb{R}^\ell.$$

It is clear that  $P(0, \xi, 0) = (0, \xi, 0)$ ,

$$P^\ell(\{0\} \times \mathbb{R}^k \times \mathbb{R}^\ell) = \{0\} \times \mathbb{R}^k \times \{0\},$$

$$\text{rank } R|_{\{1\} \times \mathbb{R}^k \times \{0\}} = k.$$

Therefore  $\text{rank } P^{k+\ell} = k$ , whence we may apply Theorem 1. The resulting smooth invariant subsystem is diffeomorphic to  $S^1 \times \mathbb{R}^k$ .

##### B. Vertical Hopper

We apply Theorem 1 to demonstrate the existence of low-dimensional invariant dynamics in the model for forced vertical hopping illustrated in Fig. 3a. The state space in the aerial phase is  $D_a := S^1 \times T\mathbb{R}_{\geq 0} \times T\mathbb{R}$ . Writing  $(\phi, x, \dot{x}, y, \dot{y}) \in D_a$ , the aerial dynamics are given in Fig. 3b. When the lower mass rests on the ground, the state space resides in  $D_g := S^1 \times T\mathbb{R}$  and the dynamics of the upper mass are obtained by restricting to the submanifold  $\{(\phi, x, \dot{x}, y, \dot{y}) : x = \dot{x} = 0\}$  as in Fig. 3b. Transition from the aerial to the ground domain occurs when the lower mass collides with the ground, and the state is reset according to  $(\phi, 0, \dot{x}, y, \dot{y}) \mapsto (\phi, y, \dot{y})$ . The lower mass lifts off when the normal force required to keep it from penetrating the ground plane becomes zero, i.e. when  $mg = -k\ell_0 - a \sin \phi + k y$ , and the state is reset via  $(\phi, y, \dot{y}) \mapsto (\phi, 0, 0, y, \dot{y})$ .

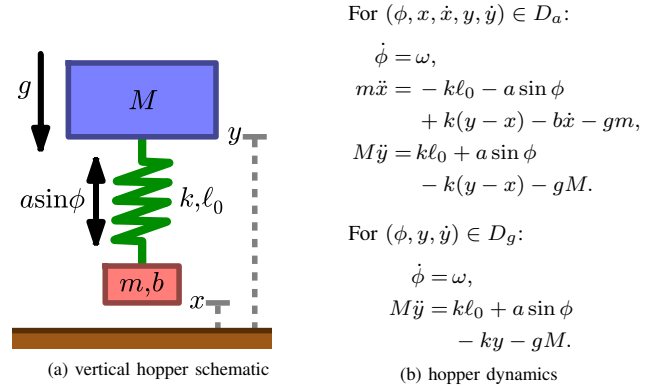


Fig. 3. Schematic and dynamics of vertical hopper. (a) Two masses  $m$  and  $M$ , constrained to move vertically above a ground plane in a gravitational field with strength  $g$ , are connected by a linear spring with stiffness  $k$  and nominal length  $\ell_0$ . The spring is equipped with an actuator that exerts a phase-varying force  $a \sin \phi$  where  $\phi = \omega$ . The lower mass experiences viscous drag proportional to velocity with constant  $b$  when it is in the air, and impacts plastically with the ground (i.e. it is not permitted to penetrate the ground and its velocity is instantaneously set to zero whenever a collision occurs). (b) Hopper dynamics in aerial and ground domains.

Numerical simulations<sup>5</sup> indicate that with parameters  $(m, M, k, b, \ell_0, a, \omega, g) = (1, 2, 10, 5, 2, 20, 2\pi, 2)$ , the hybrid system possesses a stable periodic orbit,  $\gamma$ . Choosing a Poincaré section in domain  $D_g$  at  $\phi = 0$ , we find that  $\gamma$  intersects this section at the point  $(y, \dot{y}) = (1.96, 1.88)$  and that the eigenvalues of the linearized Poincaré map are  $-0.25 \pm 0.70j$ . Both eigenvalues lie inside the unit disc, corroborating the observed stability of the orbit. Further, since neither eigenvalue is close to zero, we conclude the Poincaré map has full rank equal to 2 near its fixed point. Therefore by Remark 7 the hypotheses of Theorem 1 are satisfied, and we conclude the system's dynamics collapse to a smooth 3-dimensional subsystem after one hop.

#### V. DISCUSSION

We demonstrated the existence of a locally attracting constant-dimensional invariant subsystem near a hybrid periodic orbit whenever iterates of the associated Poincaré map have constant rank. Under a genericity condition, near a periodic orbit of a smooth dynamical system there exist *Floquet coordinates* in which the dynamics decouple into a constant-frequency phase variable and a time-invariant transverse linear system [11], [12], [13]. Under the additional rank hypothesis of Theorem 1, we obtain a canonical form for the Floquet structure of a hybrid periodic orbit. Indeed, the smooth subsystem  $(M, G)$  foliates the dynamics near the periodic orbit in each domain. Thus the behavior of the hybrid system near the orbit is a trivial extension of the behavior of a smooth system—portions of the smooth

<sup>5</sup>Note that simulation of hybrid dynamical systems is non-trivial. We make use of a recently-developed algorithm with desirable convergence properties [36]. In particular, we use Euler step size  $h = 1 \times 10^{-3}$  and relaxation parameter  $\epsilon = 1 \times 10^{-12}$ . As a note to practitioners, we found that numerical linearization of the Poincaré map via finite differences was sensitive to the coordinate displacement when using large values for the relaxation parameter. The sourcecode for this simulation is available online at <http://purl.org/sburden/cdc2011>

dynamics are “stacked” in transverse coordinates and annihilated within a finite number of cycles via a nilpotent linear operator. On the smooth subsystem, the standard construction of Floquet coordinates may be applied, generalizing the class of systems which may be analyzed using the empirical approach developed in [14], [15].

In addition to providing a canonical form for the dynamics near such non-degenerate periodic orbits, the results of this paper suggest a mechanism by which a many-legged locomotor may formally collapse a large number of degrees-of-freedom to produce a low-dimensional coordinated gait. This provides a link between currently disparate lines of research, namely the formal analysis of hybrid periodic orbits, the design of robots for locomotion and manipulation tasks, and the scientific probing of neuromechanical control architectures in organisms. It shows that hybrid models naturally exhibit dimension reduction, that this reduction may be deliberately designed into an engineered system, and that evolution may have exploited this reduction in developing its spectacular locomotors.

#### Acknowledgements & Support

We thank Saurabh Amin, Jonathan Glidden, Humberto Gonzalez, John Guckenheimer, and Ramanarayan Vasudevan for helpful conversations and careful readings of this paper.

S. Burden was supported in part by an NSF Graduate Research Fellowship. S. Revzen was supported in part by NSF Frontiers for Integrative Biology Research (FIBR), Grant No. 0425878-Neuromechanical Systems Biology. Part of this research was sponsored by the Army Research Laboratory under Cooperative Agreements W911NF-08-2-0004 and W911NF-10-2-0016.

#### REFERENCES

- [1] M. H. Dickinson, C. T. Farley, R. J. Full, M. A. R. Koehl, R. Kram, and S. Lehman. How animals move: An integrative view. *Science*, 288:100–106, 2000.
- [2] M. H. Raibert. Legged robots. *Commun. ACM*, 29(6):499–514, 1986.
- [3] U. Saranli, M. Buehler, and D.E. Koditschek. Rhex: A simple and highly mobile hexapod robot. *IJRR*, 20(7):616, 2001.
- [4] S. Kim, J.E. Clark, and M.R. Cutkosky. iSprawl: Design and tuning for high-speed autonomous open-loop running. *IJRR*, 25(9):903, 2006.
- [5] S. Grillner. Neurobiological bases of rhythmic motor acts in vertebrates. *Science*, 228:143–149, 1985.
- [6] A. Cohen, P. J. Holmes, and R. H. Rand. The nature of coupling between segmental oscillators of the lamprey spinal generator for locomotion: a model. *J Math Biol*, 13:345–369, 1982.
- [7] M. Golubitsky, I. Stewart, P. L. Buono, and J. J. Collins. Symmetry in locomotor central pattern generators and animal gaits. *Nature*, 401(6754):693–695, 1999.
- [8] L. H. Ting and J. M. Macpherson. A limited set of muscle synergies for force control during a postural task. *J Neurophysiol*, 93(1):609–613, 2005.
- [9] P. Holmes, R. J. Full, D. E. Koditschek, and J. M. Guckenheimer. The dynamics of legged locomotion: Models, analyses, and challenges. *SIAM Review*, 48(2):207–304, June 2006.
- [10] R. J. Full and D. E. Koditschek. Templates and anchors: Neuromechanical hypotheses of legged locomotion on land. *J Exp Bio*, 202:3325–3332, 1999.
- [11] G. Floquet. Sur les équations différentielles linéaires à coefficients périodiques. *Annales Scientifiques de l’École Normale Supérieure, Sér.*, 2:12, 1883.
- [12] J. Guckenheimer. Isochrons and phaseless sets. *Journal of Mathematical Biology*, 1(3):259–273, 1975.
- [13] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. Springer, 1983.
- [14] S. Revzen. *Neuromechanical Control Architectures of Arthropod Locomotion*. PhD thesis, University of California at Berkeley, 2009.
- [15] S. Revzen, J. M. Guckenheimer, and R. J. Full. Subtle differences in gaits: the perspective of data driven floquet analysis. *SICB*, Jan 2011.
- [16] T. McGeer. Passive dynamic walking. *IJRR*, 9(2):62, 1990.
- [17] J.W. Grizzle, G. Abba, and F. Plestan. Asymptotically stable walking for biped robots: Analysis via systems with impulse effects. *IEEE TAC*, 46(1):51–64, 2002.
- [18] S. Collins, A. Ruina, R. Tedrake, and M. Wisse. Efficient bipedal robots based on passive-dynamic walkers. *Science*, 307(5712):1082–1085, 2005.
- [19] R. M. Ghigliazza, R. Altendorfer, P. Holmes, and D. Koditschek. A simply stabilized running model. *SIAM Journal on Applied Dynamical Systems*, 2(2):187–218, 2003.
- [20] J.L. Proctor and P.J. Holmes. Steering by transient destabilization in piecewise-holonomic models of legged locomotion. *Regular and Chaotic Dynamics*, 13(4):267–282, 2008.
- [21] E. Klavins and D.E. Koditschek. Phase regulation of decentralized cyclic robotic systems. *IJRR*, 21(3):257, 2002.
- [22] G. C. Haynes, F. R. Cohen, and D. E. Koditschek. Gait transitions for quasi-static hexapodal locomotion on level ground. In *ISRR*, 2009.
- [23] E.R. Westervelt, J.W. Grizzle, and D.E. Koditschek. Hybrid zero dynamics of planar biped walkers. *IEEE TAC*, 48(1):42–56, 2003.
- [24] B. Morris and J.W. Grizzle. Hybrid invariant manifolds in systems with impulse effects with application to periodic locomotion in bipedal robots. *IEEE TAC*, 54(8):1751–1764, aug. 2009.
- [25] A.S. Shiriaev, L.B. Freidovich, and S.V. Gusev. Transverse linearization for controlled mechanical systems with several passive degrees of freedom. *IEEE TAC*, 55(4):893–906, 2010.
- [26] I. Poulakakis and J.W. Grizzle. The spring loaded inverted pendulum as the hybrid zero dynamics of an asymmetric hopper. *IEEE TAC*, 54(8):1779–1793, 2009.
- [27] A. Ames, R. Gregg, E. Wendel, and S. S. Sastry. On the geometric reduction of controlled three-dimensional bipedal robotic walkers. In *Workshop on Lagrangian and Hamiltonian Meth for Nonlin Ctrl*, 2006.
- [28] J.E. Marsden and T.S. Ratiu. *Introduction to mechanics and symmetry*. Springer-Verlag, 1999.
- [29] J.M. Lee. *Introduction to smooth manifolds*. Springer Verlag, 2002.
- [30] M.W. Hirsch and S. Smale. *Differential equations, dynamical systems, and linear algebra*. Academic Press, 1974.
- [31] M.W. Hirsch. *Differential topology*. Springer, 1976.
- [32] E. Wendel and A. Ames. Rank properties of poincare maps for hybrid systems with applications to bipedal walking. In *HSCC*, 2010.
- [33] A. Ames. *A Categorical Theory of Hybrid Systems*. PhD thesis, University of California at Berkeley, 2006.
- [34] F.M. Callier and C.A. Desoer. *Linear system theory*. Springer, 1991.
- [35] S. Simic, K. Johansson, J. Lygeros, and S. S. Sastry. Towards a geometric theory of hybrid systems. *Dynamics of Continuous, Discrete, and Impulsive Systems*, 12(5-6):649–687, 2005.
- [36] S. Burden, H. Gonzalez, R. Vasudevan, R. Bajcsy, and S. S. Sastry. Numerical integration of hybrid dynamical systems via domain relaxation. In *IEEE CDC*, 2011.